J Glob Optim (2011) 49:109–123
DOI 10.1007/s10898-010-9537-5

The column-sufficiency and row-sufficiency of the linear transformation on Hilbert spaces

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Received: 5 June 2009 / Accepted: 14 February 2010 / Published online: 25 February 2010
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Abstract Given a real Hilbert space H with a Jordan product and $\Omega \subset H$ being the Lorentz cone, $q \in H$, and let $T : H \to H$ be a bounded linear transformation, the corresponding linear complementarity problem is denoted by LCP(T, Ω, q). In this paper, we introduce the concepts of the column-sufficiency and row-sufficiency of T. In particular, we show that the row-sufficiency of T is equivalent to the existence of the solution of LCP(T, Ω, q) under an operator commutative condition; and that the column-sufficiency along with cross commutative property is equivalent to the convexity of the solution set of LCP(T, Ω, q). In our analysis, the properties of the Jordan product and the Lorentz cone in H are interconnected.

Keywords Linear complementarity problem · Jordan product · Lorentz cone · Column-sufficiency · Row-sufficiency

1 Introduction

Let H be a real (finite-dimensional or infinite-dimensional) Hilbert space and Ω be a closed convex cone in H. Suppose that $T : H \to H$ is a bounded linear transformation, $\langle z, w\rangle$ denotes the inner product of elements $z, w \in H$ and $\|z\|$ denotes the norm of any $z \in H$ induced by the inner product. Given $q \in H$, then the cone linear complementarity problem on H (see, for example, [3,6,9,15–17]), denoted by LCP(T, Ω, q), is to find an element $z \in H$ such that

This work was partially supported by National Nature Science Foundation of China (No. 10871144) and the Natural Science Foundation of Tianjin (No. 07JCYBJC05200).

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$$z \in \Omega, \quad Tz + q \in \Omega^*, \quad \text{and} \quad \langle z, Tz + q\rangle = 0, \tag{1.1}$$

where Ω^* is the dual cone of Ω given by $\Omega^* := \{w \in H \mid \langle w, z\rangle \geq 0, \forall z \in \Omega\}$. The cone Ω is called to be self-dual if $\Omega = \Omega^*$. If $H = R^n, \Omega = R^n_+ := \{x \in R^n \mid x \geq 0\}$, and $T = M \in R^{n\times n}$, then (1.1) reduces to the standard linear complementarity problem, denoted by LCP(M, R^n_+, q). When H is a finite-dimensional space, LCP(T, Ω, q) has been extensively studied in the literature (see, for example, [4,7,10,12–14]). In this paper, we assume that H is a general real Hilbert space with a Jordan product and Ω is the Lorentz cone in H (see the next section for the details).

A matrix $M \in R^{n\times n}$ is said to be column sufficient if for all $x \in R^n$,

$$x_i(Mx_i) \leq 0 \quad \Longrightarrow \quad x_i(Mx_i) = 0, \quad i = 1, \ldots, n$$

and row sufficient if M^T is column sufficient. They were introduced by Cottle et al. [5] in the context of LCP(M, R^n_+, q). These two concepts play very important roles in studying the solvability of LCP(M, R^n_+, q) and the convexity of the solution set of LCP(M, R^n_+, q). It was proved in [5] that M is a row sufficient matrix if and only if LCP(M, R^n_+, q) is solvable when it is feasible, and M is column sufficient if and only if the solution set of LCP(M, R^n_+, q) is convex. Gowda and Song [11] extended the column-sufficiency of a matrix to a linear transformation over the symmetric matrices space based on the convexity of the solution set of the corresponding semidefinite linear complementarity problem. Recently, Qin et al. [18] studied the column-sufficiency and row-sufficiency of a linear transformation L on a Euclidean Jordan algebra V with the help of LCP(L, K, q), where $q \in V$ and $K \subset V$ is a symmetric cone. They proved that the column-sufficiency along with the cross commutative property is equivalent to the convexity of solution set of LCP(L, K, q), and if LCP(L, K, q) has at least one solution and an operator commutative condition holds, then L is row sufficient.

In this paper, we discuss the similar results given in [18] in the setting of the general Hilbert space. We show that the row-sufficiency of T is equivalent to the existence of the solution to LCP(T, Ω, q) under an operator commutative condition; and that the column-sufficiency along with cross commutative property is equivalent to the convexity of the solution set of LCP(T, Ω, q). It should be pointed out that there three differences between [18] and this paper. Firstly, H is a general Hilbert space in this paper; while H is a finite-dimensional space in [18]. Secondly, for the row-sufficiency of T, only a necessary condition was obtained in [18]; while we will obtain a necessary and sufficient condition in this paper. Thirdly, the tools used in this paper are quite different from those in [18]. In [18], a main tool used is the Spectral Decomposition Theorem. But, such a theorem does not hold in the setting of the general Hilbert space. In our analysis, the properties of the Jordan product and the Lorentz cone in H play important roles.

This paper is organized as follows. In the next section, we introduce the basic material about the Jordan product and the Lorentz cone in H and discuss some basic properties, and then we introduce the concepts for the column-sufficiency and row-sufficiency of T. In Sect. 3, we first discuss the KKT condition of the quadratic program (QP) in H, and then establish a necessary and sufficient condition for the row-sufficiency of T. In Sect. 4, we establish a necessary and sufficient condition for the column-sufficiency of T. Some final remarks are given in Sect. 5.

2 The Lorentz cone and Jordan product in *H*

In this section, we briefly describe the Lorentz cone and the Jordan product in H, and some related results (see Refs. [3,16]. Also see Refs. [1,8,12,19] for the corresponding concepts and results given in the context of the finite-dimensional space). Moreover, several basic results are established.

For every integer $n \geq 2$, the Lorentz cone K^n in R^n can be written as

$$K^n = \{z = (r, x) \in R \times R^{n-1} \mid r \geq \|x\|\}.$$

Let $e = (1, 0) \in R \times R^{n-1}$, then $\langle z, e\rangle = r$. Hence, $K^n = \left\{z \in R^n \mid \langle z, e\rangle \geq \frac{1}{\sqrt{2}}\|z\|\right\}$. Consider the following closed convex cone in the real Hilbert space H:

$$\Omega(e, r) = \{z \in H \mid \langle z, e\rangle \geq r\|z\|\},$$

where $0 < r < 1$ and $e \in H$ with $\|e\| = 1$. It is easy to prove that $\Omega(e, r)$ is pointed, i.e., $\Omega(e, r) \bigcap (-\Omega(e, r)) = \{0\}$. Define the orthogonal complement of e by

$$< e >^{\perp} := \{z \in H \mid \langle z, e\rangle = 0\}.$$

For any $z \in H$, we have the orthogonal decomposition $z = x + \lambda e$ with unique $x \in < e >^{\perp}$ and $\lambda \in R$ (in fact, $\lambda = \langle z, e\rangle$). Thus,

$$\Omega(e, r) = \left\{x + \lambda e \mid x \in < e >^{\perp} \text{ and } \lambda \in R \text{ with } \lambda \geq \frac{r}{\sqrt{1-r^2}}\|x\|\right\}.$$

Proposition 2.1 ([3, Proposition 2.1]) *For any $e \in H$ with $\|e\| = 1$ and $0 < r < 1$, the dual cone of $\Omega(e, r)$ can be written as*

$$\Omega^*(e, r) = \{z \in H \mid \langle z, w\rangle \geq 0 \text{ for all } w \in \Omega(e, r)\} = \Omega\left(e, \sqrt{1-r^2}\right).$$

Consequently, $\Omega\left(e, \frac{1}{\sqrt{2}}\right)$ is a self-dual closed convex cone.

Proof For completeness, we give the proof of [3, Proposition 2.1] as follows. Let $z = x + \lambda e \in \Omega(e, \sqrt{1-r^2})$ and $w = y + \mu e \in \Omega(e, r)$ be arbitrary. Since $\lambda\mu \geq \|x\|\|y\|$, we have $\langle z, w\rangle \geq \langle x, y\rangle + \|x\|\|y\| \geq 0$. This proves that $\Omega(e, \sqrt{1-r^2}) \subseteq \Omega^*(e, r)$.

Conversely, we prove that if $z = x + \lambda e \in \Omega^*(e, r)$, then $x \in \Omega(e, \sqrt{1-r^2})$, i.e., $\lambda \geq r^{-1}\sqrt{1-r^2}\|x\|$. This is trivial when $x = 0$. When $x \neq 0$, by considering the element $v = -r^{-1}\sqrt{1-r^2}x + \|x\|e$ of $\Omega(e, r)$, we have

$$0 \leq \langle x, v\rangle = -r^{-1}\sqrt{1-r^2}\|x\|^2 + \lambda\|x\|.$$

The proof is complete. □

From the above proposition, $\Omega\left(e, \frac{1}{\sqrt{2}}\right)$ can be written as

$$\Omega\left(e, \frac{1}{\sqrt{2}}\right) = \{x + \lambda e \in H \mid x \in < e >^{\perp} \text{ and } \lambda \in R \text{ with } \lambda \geq \|x\|\}. \tag{2.1}$$

When $H = R^n$ and $e = (1, 0) \in R \times R^{n-1}$, the set $\Omega\left(e, \frac{1}{\sqrt{2}}\right)$ coincides with the Lorentz cone K^n in R^n. Hence, $\Omega\left(e, \frac{1}{\sqrt{2}}\right)$ is called the Lorentz cone in the Hilbert space H determined by e. In this paper, we simply denote $\Omega\left(e, \frac{1}{\sqrt{2}}\right)$ by Ω.

In the following, we introduce the concept of Jordan product in the Hilbert space H and some related conclusions.

For any $z, w \in H$ with $z = x + \lambda e$ and $w = y + \mu e$, where $x, y \in \,<e>^{\perp}$ and $\lambda, \mu \in R$, **the Jordan product** $z \circ w$ of z and w is defined by

$$z \circ w = \mu x + \lambda y + \langle z, w\rangle e = \mu x + \lambda y + (\langle x, y\rangle + \lambda\mu)e. \tag{2.2}$$

Denote $z^2 = z \circ z$ for any $z \in H$. The following properties are easily verified.

- $z \circ w = w \circ z$ and $z \circ e = z$ for any $z, w \in H$.
- $(z + w) \circ v = z \circ v + w \circ v$ for any $z, w, v \in H$.
- $\langle z, w \circ v\rangle = \langle w, z \circ v\rangle = \langle v, z \circ w\rangle$ for any $z, w, v \in H$.
- $z \circ (z^2 \circ w) = z^2 \circ (z \circ w)$ for any $z, w \in H$.
- $z^2 = 2\lambda x + \|z\|^2 e \in \Omega$ for any $z = x + \lambda e \in H$ with $x \in \,<e>^{\perp}$ and $\lambda \in R$.

Note that the Jordan product is not associative even in the finite-dimensional Euclidean space. In fact, it is easy to show that Ω is the cone of squares w.r.t. $\circ$ multiplication. After introducing the Jordan product, the space H becomes a Jordan algebra (see [8]). For any $z, w \in H$, we say that z and w *operator commute* if $z \circ (w \circ u) = w \circ (z \circ u)$ holds for any $u \in H$.

The following several lemmas give the conditions and properties of z and w operator commuting.

Lemma 2.1 *For any $z, w \in H$ with $z = x + \lambda e$, $w = y + \mu e$, where $x, y \in \,<e>^{\perp}$ and $\lambda, \mu \in R$, then z and w operator commute if and only if there is an $\alpha \in R$ (possibly depends on x and y) such that $y = \alpha x$ or $x = \alpha y$. In particular, if $x \neq 0$ (respectively, $y \neq 0$), then z and w operator commute if and only if there is an $\alpha \in R$ such that $y = \alpha x$ (respectively, $x = \alpha y$).*

Proof If $x = 0$ or $y = 0$, then the results of this lemma hold obviously. In the following, we assume that $x \neq 0$ and show that z and w operator commute if and only if there is an $\alpha \in R$ such that $y = \alpha x$. The case of $y \neq 0$ can be discussed similarly. By using the definition of z and w operator commuting, we need to show that $z \circ (w \circ u) = w \circ (z \circ u)$ holds for any $u \in H$ if and only if there is an $\alpha \in R$ such that $y = \alpha x$. Let $u = v + \tau e$ where $v \in \,<e>^{\perp}$ and $\tau \in R$, we have

$$\begin{aligned} z \circ (w \circ u) &= (x + \lambda e) \circ ((\tau y + \mu v) + (\langle y, v\rangle + \tau\mu)e) \\ &= \tau\lambda y + \lambda\mu v + \langle y, v\rangle x + \tau\mu x + \tau\langle x, y\rangle e + \mu\langle x, v\rangle e + \lambda\langle y, v\rangle e + \tau\lambda\mu e \end{aligned}$$

and

$$\begin{aligned} w \circ (z \circ u) &= (y + \mu e) \circ ((\tau x + \lambda v) + (\langle x, v\rangle + \tau\lambda)e) \\ &= \tau\mu x + \lambda\mu v + \langle x, v\rangle y + \tau\lambda y + \tau\langle x, y\rangle e + \lambda\langle y, v\rangle e + \mu\langle x, v\rangle e + \tau\lambda\mu e. \end{aligned}$$

These imply that

$$z \circ (w \circ u) = w \circ (z \circ u) \quad \text{if and only if} \quad \langle y, v\rangle x = \langle x, v\rangle y. \tag{2.3}$$

On one hand, suppose that $z \circ (w \circ u) = w \circ (z \circ u)$ holds for any $u \in H$. Since $x \neq 0$ and v is arbitrary, we may choose $v \in H$ such that $\langle x, v\rangle \neq 0$. Thus, by taking $\alpha := \frac{\langle y, v\rangle}{\langle x, v\rangle}$, we obtain from (2.3) that there is an $\alpha \in R$ such that $y = \alpha x$. On the other hand, suppose that there is an $\alpha \in R$ such that $y = \alpha x$. Then,

$$\langle y, v\rangle x = \langle \alpha x, v\rangle x = \alpha\langle x, v\rangle x = \langle x, v\rangle \alpha x = \langle x, v\rangle y$$

holds for any $v \in H$. Thus, by (2.3) it follows that $z \circ (w \circ u) = w \circ (z \circ u)$ for any $u \in H$. Therefore, we complete the proof of this lemma. □

Remark In the setting of the finite-dimensional space, such a result has been obtained. For example, see [1] for the one in the case of $H = R^n$.

Lemma 2.2 *Let $z = x + \lambda e$, $w = y + \mu e \in H$ with $x, y \in \langle e \rangle^\perp$ and $\lambda, \mu \in R$.*

(i) The following two conditions are equivalent:

(a) $z \in \Omega$, $w \in \Omega$, and $\langle z, w\rangle = 0$;
(b) $z \in \Omega$, $w \in \Omega$, and $z \circ w = 0$.

In each case, we may get that z and w operator commute.

(ii) Suppose that $x \neq 0$ and $y \neq 0$, and one of (a) and (b) in (i) holds. Then,

(c) $\lambda > 0$ and $\mu > 0$;
(d) if there exists real number α such that $y = \alpha x$, then $\alpha < 0$;
(e) $\|x\| = \lambda$ and $\|y\| = \mu$.

Proof Firstly, we show that the result (i) holds. Let $z = x + \lambda e$ and $w = y + \mu e$. Since $z \circ w = \mu x + \lambda y + \langle z, w\rangle e$, it is easy to verify that (b) implies (a). Next, we prove that (a) implies (b). Because $z \in \Omega, w \in \Omega$ and $\langle z, w\rangle = \langle x, y\rangle + \lambda\mu = 0$, we have $z \circ w = \mu x + \lambda y$, $\lambda > \|x\|$ and $\mu > \|y\|$. Hence,

$$\begin{aligned} \|\mu x + \lambda y\|^2 &= \mu^2\|x\|^2 + 2\lambda\mu\langle x, y\rangle + \lambda^2\|y\|^2 \\ &= \mu^2\|x\|^2 - 2\lambda^2\mu^2 + \lambda^2\|y\|^2 = 0. \end{aligned}$$

It follows that $\mu x + \lambda y = 0$, i.e., $z \circ w = \mu x + \lambda y = 0$. This proves that (a) implies (b).

Secondly, we show that the results in (ii) hold. Since $z \in \Omega$ and $w \in \Omega$, it follows from the definition of Ω that

$$\|x\| \leq \lambda \quad \text{and} \quad \|y\| \leq \mu. \tag{2.4}$$

This, together with $x \neq 0$ and $y \neq 0$, implies that the results given in (c) hold. Now, for part (d). From (i) we know that z and w operator commute. Thus, it follows from Lemma 2.1 that there is a real number α such that $y = \alpha x$. Since $\langle z, w\rangle = 0$, we have

$$0 = \langle z, w\rangle = \langle x + \lambda e, y + \mu e\rangle = \langle x, y\rangle + \lambda\mu = \langle x, \alpha x\rangle + \lambda\mu = \alpha\|x\|^2 + \lambda\mu, \tag{2.5}$$

which demonstrates $\alpha < 0$, and hence, the result in (d) holds. By (2.5) we have

$$\lambda\mu = -\alpha\|x\|^2 = -\alpha\|x\|\|x\| = \|x\|\|\alpha x\| = \|x\|\|y\|. \tag{2.6}$$

By (2.4) and (2.6), part (e) holds true. □

Remark In the setting of the finite-dimensional space, the result given in the case (i) of this lemma has been obtained. For example, see [12] for the one in the case of H being a Euclidean Jordan algebra.

Lemma 2.3 *Given $u = v + \xi e \in H$ with $0 \neq v \in \langle e \rangle^\perp$ and $\xi \in R$, for any $z, w \in H$, if both z and w operator commute with u, then z and w operator commute.*

Proof Let $z = x + \lambda e$ and $w = y + \mu e$ with $x, y \in \langle e \rangle^\perp$ and $\lambda, \mu \in R$. We consider the following two cases:

Case 1 Suppose that $x = 0$ or $y = 0$. Since λe (or μe) operator commutes with an arbitrary element in H, it is easy to see that z and w operator commute in this case.

Case 2 Suppose that $x \neq 0$ and $y \neq 0$. Since both z and w operator commute with u and $v \neq 0$, it follows from Lemma 2.1 that there exist $\alpha, \beta \in R$ such that $x = \alpha v$ and $v = \beta y$. Thus, $x = \alpha\beta y$. Again, by Lemma 2.1, we conclude that z and w operator commute.

The proof is complete. □

Lemma 2.4 *For any $z, w \in \Omega$, if z and w operator commute, then $z \circ w \in \Omega$.*

Proof Let $z = x + \lambda e$ and $w = y + \mu e$ with $x, y \in <e>^{\perp}$ and $\lambda, \mu \in R$. By using the condition $z, w \in \Omega$ and the definition of Ω, we have $\lambda \geq \|x\|$ and $\mu \geq \|y\|$. Since z operator commutes with w, it follows from Lemma 2.1 that there exists an $\alpha \in R$ such that $y = \alpha x$ or $x = \alpha y$. Without loss of generality, let $y = \alpha x$. Then,

$$\begin{aligned}\langle x, y\rangle + \lambda\mu - \|\mu x + \lambda y\| &= \alpha\|x\|^2 + \lambda\mu - |\mu + \alpha\lambda|\|x\| \\ &= \begin{cases} (\lambda - \|x\|)(\mu - \alpha\|x\|), & \text{when } \mu + \alpha\lambda \geq 0, \\ (\lambda + \|x\|)(\mu + \alpha\|x\|), & \text{when } \mu + \alpha\lambda < 0. \end{cases}\end{aligned}$$

Since $\lambda \geq \|x\|$ and $\mu \geq \|y\| = |\alpha|\|x\|$, we obtain that $\langle x, y\rangle + \lambda\mu \geq \|\mu x + \lambda y\|$. By (2.2) and the definition of Ω, we obtain that $z \circ w \in \Omega$. □

Lemma 2.5 *Given $z, w, u \in H$, if z and w operator commute and z and $\tau w + \eta u$ operator commute where τ and η are two non-zero real numbers, then z and u operator commute.*

Proof For any $v \in H$, since z and w operator commute, we have

$$z \circ (w \circ v) = w \circ (z \circ v); \tag{2.7}$$

and since z and $\tau w + \eta u$ operator commute, we have

$$z \circ ((\tau w + \eta u) \circ v) = (\tau w + \eta u) \circ (z \circ v),$$

i.e.,

$$\tau z \circ (w \circ v) + \eta z \circ (u \circ v) = \tau w \circ (z \circ v) + \eta u \circ (z \circ v). \tag{2.8}$$

By combining (2.7) with (2.8), we obtain that $z \circ (u \circ v) = w \circ (z \circ v)$ holds for any $v \in H$. Thus, z and u operator commute. □

In the rest of this paper, we consider LCP(T, Ω, q):

$$z \in \Omega, \quad Tz + q \in \Omega, \quad \text{and} \quad \langle z, Tz + q\rangle = 0, \tag{2.9}$$

where H is a general Hilbert space with the Jordan product being defined by (2.2), $\Omega \subset H$ is the Lorentz cone $\Omega\left(e, \frac{1}{\sqrt{2}}\right)$ defined by (2.1), $T : H \to H$ is a bounded linear transformation, and $q \in H$. We always assume that e is an arbitrary given element in H with $\|e\| = 1$.

Now, we introduce the concepts of the column-sufficiency and row-sufficiency of T. Let T^* denote the *adjoint operator* of the bounded linear transformation T, i.e., T^* satisfies $\langle z, Tw\rangle = \langle T^*z, w\rangle$ for all $z, w \in H$. Then, T^* is also a bounded linear transformation on H.

Definition 2.1 A bounded linear transformation $T : H \to H$ is

- column sufficient if

$$\left.\begin{array}{l} z \text{ and } Tz \text{ operator commute} \\ -z \circ Tz \in \Omega \end{array}\right\} \implies z \circ Tz = 0;$$

- row sufficient if T^* is column sufficient.

We will investigate properties of the row-sufficiency and the column-sufficiency of T in the following two sections.

3 The row-sufficiency of T

In this section, we discuss the relations between the row-sufficiency of T and the solution of LCP(T, Ω, q) with the help of the KKT condition of the following quadratic programming over Ω (denoted by (QP)):

$$\begin{array}{ll} \min & \frac{1}{2}\langle z, Tz + T^*z\rangle + \langle q, z\rangle \\ \text{s.t.} & z \in \Omega, Tz + q \in \Omega. \end{array}$$

Theorem 3.1 *The KKT condition of (QP) has the following form:*

$$z^* \in \Omega, \quad q + Tz^* + T^*z^* - T^*w^* \in \Omega, \tag{3.1}$$

$$\langle z^*, q + Tz^* + T^*z^* - T^*w^*\rangle = 0, \tag{3.2}$$

$$w^* \in \Omega, \quad Tz^* + q \in \Omega, \tag{3.3}$$

$$\langle w^*, Tz^* + q\rangle = 0. \tag{3.4}$$

Furthermore, if $z^* - w^*$ and $T^*(z^* - w^*)$ operator commute, then (z^*, w^*) satisfies

$$(w^* - z^*) \circ T^*(z^* - w^*) \in \Omega. \tag{3.5}$$

Proof For the quadratic programming (QP), define $f(z) := \frac{1}{2}\langle z, Tz + T^*z\rangle + \langle q, z\rangle$ and $h(z) := Tz + q$. Then the KKT condition (the optimality condition) for (QP) can be written as (see [2]):

$$0 \in \partial_z L(z^*, w^*) + N_\Omega(z^*), \quad -w^* \in N_\Omega(h(z^*)), \tag{3.6}$$

where $L(z, w) := f(z) - \langle w, h(z)\rangle$, $(z, w) \in \Omega \times \Omega^*$, $\partial_x L(\cdot)$ denotes the subdifferential of the function L, and $N_\Omega(z) := \{w \in H | \langle w, u - z\rangle \leq 0, \ \forall u \in \Omega\}$ is the normal cone of Ω at z. By using the convexity and self-duality of Ω, (3.6) becomes

$$\begin{aligned} z^* \in \Omega, \quad \partial_z L(z^*, w^*) \in \Omega, \quad \langle z^*, \partial_z L(z^*, w^*)\rangle = 0; \\ h(z^*) \in \Omega, \quad w^* \in \Omega, \quad \langle w^*, h(z^*)\rangle = 0. \end{aligned}$$

This implies that conditions (3.1)–(3.4) hold.

In order to show that (3.5) holds, we need to prove that the following result holds:

$$\langle z^* - w^*, T^*(z^* - w^*)\rangle \leq 0. \tag{3.7}$$

In fact, by (3.1)–(3.4), we have

$$\langle z^*, q + Tz^*\rangle = -\langle z^*, T^*z^* - T^*w^*\rangle \geq 0$$

and

$$0 \leq \langle w^*, q + Tz^* + T^*z^* - T^*w^* \rangle = \langle w^*, T^*z^* - T^*w^* \rangle.$$

Hence, $\langle z^* - w^*, T^*(z^* - w^*) \rangle = \langle z^*, T^*z^* - T^*w^* \rangle - \langle w^*, q + Tz^* + T^*z^* - T^*w^* \rangle \leq 0$, i.e., (3.7) holds.

In addition, by (3.1), (3.2), and Lemma 2.2 (i), we have

$$z^* \circ (q + Tz^* + T^*(z^* - w^*)) = 0 \tag{3.8}$$

and z^* operator commutes with $q + Tz^* + T^*(z^* - w^*)$. Similarly, by (3.3), (3.4), and Lemma 2.2 (i), we have

$$w^* \circ (Tz^* + q) = 0 \tag{3.9}$$

and w^* operator commutes with $Tz^* + q$. Let

$$z^* = x + \lambda e, \quad w^* = y + \mu e, \quad Tz^* + q = u + \xi e, \quad \text{and} \quad T^*(z^* - w^*) = v + \zeta e,$$

where $x, y, u, v \in < e >^{\perp}$ and $\lambda, \mu, \xi, \zeta \in R$.

Now, we prove that (3.5) holds, which is divided into three parts:

- Suppose that $y = 0$. In this case, $w^* = \mu e$, and hence, it is obvious that w^* and $q + Tz^* + T^*(z^* - w^*)$ operator commute. By (3.1), (3.3) and Lemma 2.4, it follows that $w^* \circ (q + Tz^* + T^*(z^* - w^*)) \in \Omega$, which and (3.9) lead to

  $$w^* \circ T^*(z^* - w^*) \in \Omega.$$

  Next, we show that $-z^* \circ T^*(z^* - w^*) \in \Omega$. Consider the following two cases:

  - If $\mu = 0$, then $w^* = 0$. Since z^* operator commutes with $q + Tz^* + T^*z^*$ and z^* operator commutes with T^*z^*, by Lemma 2.5 we have that z^* operator commutes with $q + Tz^*$. This, together with (3.1) and (3.3) with Lemma 2.4, implies that $z^* \circ (Tz^* + q) \in \Omega$. Thus, it follows that $-z^* \circ T^*(z^* - w^*) \in \Omega$ from (3.8).
  - If $\mu \neq 0$, then by (3.4), we have that $0 = \langle w^*, Tz^* + q \rangle = \langle \mu e, u + \xi e \rangle = \mu\xi$. Hence, it follows that $\xi = 0$, and $u = 0$ by $0 \leq \|u\| \leq \xi$. Thus, $Tz^* + q = 0$. Furthermore, $-z^* \circ T^*(z^* - w^*) = 0 \in \Omega$ by (3.8).

  Therefore, from the definition of a cone, it follows that $(w^* - z^*) \circ T^*(z^* - w^*) \in \Omega$.
- Suppose that $x = 0$. In this case, we discuss the cases either $\lambda = 0$ or $\lambda \neq 0$.

  - When $\lambda = 0$, i.e., $z^* = 0$, By (3.3) and (3.4), we have that w^* operator commutes with q. Since $z^* - w^*$ and $T^*(z^* - w^*)$ operator commute, we get that w^* operator commutes with $T^*(w^*)$. Thus, w^* and $q + Tz^* + T^*(z^* - w^*)$ operator commute. By (3.1), (3.3) and Lemma 2.4, it follows that $w^* \circ (q + Tz^* + T^*(z^* - w^*)) \in \Omega$, which and (3.9) lead to $w^* \circ T^*(z^* - w^*) \in \Omega$. By $z^* \circ T^*(z^* - w^*) = 0 \in \Omega$, we have $(w^* - z^*) \circ T^*(z^* - w^*) \in \Omega$.
  - When $\lambda \neq 0$, we get that $z^* = \lambda e$. By (3.1) and (3.2), it is easy to verify that $q + Tz^* + T^*(z^* - w^*) = 0$. Thus, $w^* \circ (q + Tz^* + T^*(z^* - w^*)) = 0$. By (3.9), it follows that $w^* \circ T^*(z^* - w^*) = 0$. Since $z^* = \lambda e$, z^* and $Tz^* + q$ operator commute. By (3.1), (3.3) and Lemma 2.4, we have $z^* \circ (Tz^* + q) \in \Omega$. Together with (3.8), it implies that $-z^* \circ T^*(z^* - w^*) \in \Omega$. Hence, $(w^* - z^*) \circ T^*(z^* - w^*) \in \Omega$.

- Suppose that $x \neq 0$ and $y \neq 0$. we consider the following two cases:
  - When $x = y$, since w^* operator commutes with $Tz^* + q$, we have that z^* and $Tz^* + q$ operator commute. By (3.1), (3.3) and Lemma 2.4, we conclude that $z^* \circ (Tz^* + q) \in \Omega$. So, from (3.8) it follows that $-z^* \circ T^*(z^* - w^*) \in \Omega$. Similarly, since z^* operator commutes with $q + Tz^* + T^*(z^* - w^*)$, we have that w^* and $q + Tz^* + T^*(z^* - w^*)$ operator commute. By (3.1), (3.3) and Lemma 2.4, we conclude that $w^* \circ (q + Tz^* + T^*(z^* - w^*)) \in \Omega$. This, together with (3.9), implies that $w^* \circ T^*(z^* - w^*) \in \Omega$. Hence, by the definition of a cone, we have $(w^* - z^*) \circ T^*(z^* - w^*) \in \Omega$.
  - When $x \neq y$, since z^* operator commutes with $q + Tz^* + T^*(z^* - w^*)$ and w^* operator commutes with $Tz^* + q$, by Lemma 2.1, it follows that there are $\alpha, \beta \in R$ such that

$$u + v = \alpha x \quad \text{and} \quad u = \beta y. \tag{3.10}$$

Since $z^* - w^*$ and $T^*(z^* - w^*)$ operator commute, there exists $\gamma \in R$ such that $v = \gamma(x - y)$. This and (3.10) yield

$$(\beta - \gamma)y = (\alpha - \gamma)x. \tag{3.11}$$

(A) Suppose that $\beta - \gamma \neq 0$ or $\alpha - \gamma \neq 0$. Then, we obtain from (3.11) that $y = \frac{\alpha-\gamma}{\beta-\gamma}x$ or $x = \frac{\beta-\gamma}{\alpha-\gamma}y$. This implies by Lemma 2.1 that z^* and w^* operator commute. Because z^* and $q + Tz^* + T^*(z^* - w^*)$ operator commute, by Lemma 2.3 we get that w^* operator commutes with $q + Tz^* + T^*(z^* - w^*)$, which, together with (3.1), (3.3) and Lemma 2.4, implies that $w^* \circ (q + Tz^* + T^*(z^* - w^*)) \in \Omega$. Thus, it follows from (3.9) that $w^* \circ T^*(z^* - w^*) \in \Omega$. In addition, since z^* and w^* operator commute; and w^* and $q + Tz^*$ operator commute, it follows from Lemma 2.3 that z^* and $q + Tz^*$ operator commute. Thus, by (3.1), (3.3) and Lemma 2.4, we obtain that $z^* \circ (q + Tz^*) \in \Omega$. This, together with (3.8), implies that $-z^* \circ T^*(z^* - w^*) \in \Omega$. Hence, we have $(w^* - z^*) \circ T^*(z^* - w^*) \in \Omega$.

(B) Suppose that one of $\beta - \gamma$ and $\alpha - \gamma$ is zero. Then, both of them are zeros since $x \neq 0$ and $y \neq 0$. Thus, $\gamma = \alpha = \beta$.
If $\alpha = 0$, then $u + v = 0$ and $u = v = 0$ from (3.10), and hence, $q + Tz^* + T^*(z^* - w^*) = (\xi + \zeta)e$. When $(\xi + \zeta) = 0$, then $\xi = -\zeta$. If $\zeta = 0$, it follows that $T^*(z^* - w^*) = 0$, i.e., we have $(w^* - z^*) \circ T^*(z^* - w^*) \in \Omega$. If $\xi = -\zeta \neq 0$, then $Tz^* + q = \xi e \neq 0$. Thus, it follows from (3.3) and (3.4) that $w^* = y + \mu e = 0$, which is a contradiction with $y \neq 0$. This demonstrates that this case can not happen. When $(\xi + \zeta) \neq 0$, it follows from (3.1) and (3.2) that $z^* = 0$, which is a contradiction with $x \neq 0$. This demonstrates that this case cannot happen.
If $\alpha \neq 0$, by the results (d) and (e) in Lemma 2.2 (ii) we have

(1) $\gamma = \alpha = \beta < 0$;
(2) $\|x\| = \lambda$, $\|y\| = \mu$, $\|u\| = \xi$, $\|u + v\| = \xi + \zeta$.

By combining (3.10) with the above results (1) and (2), we have that $\alpha = -\frac{\xi+\zeta}{\lambda}$, $\beta = -\frac{\xi}{\mu}$, and

$$\frac{\xi + \zeta}{\lambda} = \frac{\xi}{\mu} \quad \text{i.e.,} \quad (\lambda - \mu)\xi = \mu\zeta. \tag{3.12}$$

By the definition of the Jordan product,

$$(w^* - z^*) \circ T^*(z^* - w^*) = \zeta(y - x) + (\mu - \lambda)v + \langle w^* - z^*, T^*(z^* - w^*)\rangle e,$$

and

$$\begin{aligned}\zeta(y - x) + (\mu - \lambda)v &= \zeta(y - x) - \gamma(\mu - \lambda)(y - x)\\ &= \left(\zeta + \frac{(\mu - \lambda)\xi}{\mu}\right)(y - x)\\ &= (\zeta - \zeta)(y - x) = 0\end{aligned}$$

where the second equality follows from the result (1) and the third equality follows from (3.12). By (3.7) and the definition of Ω, we conclude that $(w^* - z^*) \circ T^*(z^* - w^*) \in \Omega$.

Therefore, we obtain that $(w^* - z^*) \circ T^*(z^* - w^*) \in \Omega$. The proof is complete. □

Remark If (z^*, w^*) satisfies (3.1)–(3.4), then it is called a KKT point of (QP).

Theorem 3.2 *Given a bounded linear transformation $T : H \to H$, the following conditions are equivalent:*

- (a) *T is the row sufficient.*
- (b) *For any given $q \in H$, if the KKT point (z, w) of (QP) satisfies that $z - w$ and $T^*(z - w)$ operator commute, then z is a solution of LCP(T, Ω, q).*

Proof $(a) \Rightarrow (b)$: Suppose that (z, w) is a KKT point of (QP) and $z - w$ operator commutes with $T^*(z - w)$. By Theorem 3.1, we get that (z, w) satisfies the formulas (3.1)–(3.4) and $-(z - w) \circ T^*(z - w) \in \Omega$. Since T is the row sufficient, we have $(z - w) \circ T^*(z - w) = 0$. Hence, $\langle z - w, T^*(z - w)\rangle = 0$, i.e.,

$$\langle z, T^*(z - w)\rangle = \langle w, T^*(z - w)\rangle. \tag{3.13}$$

By (3.2), we get

$$\langle z, Tz + q\rangle + \langle z, T^*(z - w)\rangle = 0. \tag{3.14}$$

Since Ω is a self-dual cone, we have $\langle w, q + Tz + T^*(z - w)\rangle \geq 0$ from (3.1) and (3.3). This, together with (3.4), leads to $\langle w, T^*(z - w)\rangle \geq 0$. Combining (3.13) and (3.14), we get $\langle z, Tz + q\rangle \leq 0$. However, $z \in \Omega$ and $Tz + q \in \Omega$ imply that $\langle z, Tz + q\rangle \geq 0$. Thus, $\langle z, Tz + q\rangle = 0$. This demonstrates that z is a solution of LCP(T, Ω, q).

$(b) \Rightarrow (a)$: Suppose that there exists $z \in H$ such that z and T^*z operator commute and $-z \circ T^*z \in \Omega$, but $z \circ T^*z \neq 0$. Let $z = x + \lambda e$ and $T^*z = y + \mu e$ with $x, y \in < e >^{\perp}$ and $\lambda, \mu \in R$. Since z and T^*z operator commute, there exists $\alpha \in R$ such that $y = \alpha x$ or $x = \alpha y$. Without loss of generality, let $y = \alpha x$. We consider two cases:

Case 1 If $x = 0$, then $z = \lambda e$, $T^*z = \mu e$ and $-z \circ T^*z = -\lambda\mu e \in \Omega$. By $z \circ T^*z \neq 0$, we obtain that $\lambda\mu < 0$.

- Suppose that $\lambda > 0$, then $\mu < 0$. Let $q := -Tz - \mu e$. Then, $q + Tz + T^*(z - 0) = -\mu e + T^*z = 0 \in \Omega$. It is easy to verify that $(z, 0)$ satisfies that the formulas (3.1)–(3.4) and z operator commutes with T^*z. This means that the conditions given in (b) hold. Thus, $z = \lambda e$ solves LCP(T, Ω, q). This implies that $0 = \langle z, Tz + q\rangle = \langle z, Tz + (-Tz - \mu e)\rangle = -\lambda\mu$, which is a contradiction with $\lambda\mu < 0$.

- Suppose that $\lambda < 0$. Let $q := Tz + \mu e$. In a similar way as the above, we can get that $(-z, 0)$ satisfies the formulas (3.1)–(3.4) and $-z$ operator commutes with $T^*(-z)$. Thus, $-z$ is a solution of LCP(T, Ω, q). This implies that $0 = \langle -z, T(-z) + q\rangle = -\lambda\mu$, which is a contradiction with $\lambda\mu < 0$.

These demonstrate that T is the row sufficient.

Case 2 If $x \neq 0$, we let $e_1 = \frac{e - \frac{x}{\|x\|}}{2}$ and $e_2 = \frac{e + \frac{x}{\|x\|}}{2}$. Then,

$$z = x + \lambda e = (\lambda - \|x\|)e_1 + (\lambda + \|x\|)e_2, \quad \langle e_1, e_2\rangle = 0, \quad \text{and} \quad e_1, e_2 \in \Omega.$$

By Lemma 2.2 (i), we have $e_1 \circ e_2 = 0$. Since $y = \alpha x$ and $e_1 + e_2 = e$, it follows that

$$T^* z = y + \mu e = (\mu - \alpha\|x\|)e_1 + (\mu + \alpha\|x\|)e_2.$$

Thus,

$$\begin{aligned} -z \circ T^* z &= -(\lambda - \|x\|)(\mu - \alpha\|x\|)e_1 - (\lambda + \|x\|)(\mu + \alpha\|x\|)e_2 \\ &= -(\mu + \lambda\alpha)x - (\alpha\|x\|^2 + \lambda\mu)e. \end{aligned}$$

By using $-z \circ T^* z \in \Omega$, self-duality of Ω, and the representation of $-z \circ T^* z$ in e_1 and e_2 given in the above line, we obtain that either $(\lambda - \|x\|)(\mu - \alpha\|x\|) \leq 0$ and $(\lambda + \|x\|)(\mu + \alpha\|x\|) < 0$ or $(\lambda - \|x\|)(\mu - \alpha\|x\|) < 0$ and $(\lambda + \|x\|)(\mu + \alpha\|x\|) \leq 0$. Without loss of generality, we consider only the first case. Let

$$\begin{aligned} z^+ &:= \max\{\lambda - \|x\|, 0\}e_1 + \max\{\lambda + \|x\|, 0\}e_2; \\ z^- &:= \max\{\|x\| - \lambda, 0\}e_1 + \max\{-\lambda - \|x\|, 0\}e_2; \\ (T^* z)^+ &:= \max\{\mu - \alpha\|x\|, 0\}e_1 + \max\{\mu + \alpha\|x\|, 0\}e_2; \\ (T^* z)^- &:= \max\{\alpha\|x\| - \mu, 0\}e_1 + \max\{-\mu - \alpha\|x\|, 0\}e_2. \end{aligned}$$

Then $z = z^+ - z^-$ and $T^* z = (T^* z)^+ - (T^* z)^-$.

- Consider the case of $z^+ \neq 0$. In this case, we define $q = -Tz^+ + (T^* z)^-$. It is easy to prove that $z^+, z^- \in \Omega$, $Tz^+ + q = (T^* z)^- \in \Omega$ and $q + Tz^+ + T^* z^+ - T^* z^- = (T^* z)^+ \in \Omega$. Furthermore,

$$\begin{aligned} \langle z^+, q + Tz^+ + T^* z^+ - T^* z^-\rangle &= \langle z^+, (T^* z)^+\rangle \\ &= \frac{1}{2}\max\{\lambda - \|x\|, 0\}\max\{\mu - \alpha\|x\|, 0\} \\ &\quad + \frac{1}{2}\max\{\lambda + \|x\|, 0\}\max\{\mu + \alpha\|x\|, 0\} \\ &= 0, \end{aligned}$$

where the last equality holds is due to that $(\lambda - \|x\|)(\mu - \alpha\|x\|) \leq 0$ and $(\lambda + \|x\|)(\mu + \alpha\|x\|) < 0$. Similarly, we obtain that

$$\begin{aligned} \langle z^-, Tz^+ + q\rangle &= \langle z^-, (T^* z)^-\rangle \\ &= \frac{1}{2}\max\{\|x\| - \lambda, 0\}\max\{\alpha\|x\| - \mu, 0\} \\ &\quad + \frac{1}{2}\max\{\|x\| - \lambda, 0\}\max\{\alpha\|x\| - \mu, 0\} \\ &= 0. \end{aligned}$$

By Theorem 3.1, it follows that (z^+, z^-) is a KKT point of (QP). Moreover, it is easy to prove that $z = z^+ - z^-$ operator commutes with $T^* z = T^*(z^+ - z^-)$. By the condition (b), we conclude that z^+ is a solution of LCP(T, Ω, q) and

$$\begin{aligned} 0 &= \langle z^+, Tz^+ + q\rangle = \langle z^+, (T^*z)^-\rangle \\ &= \frac{1}{2}\max\{\lambda - \|x\|, 0\}\max\{\alpha\|x\| - \mu, 0\} \\ &\quad + \frac{1}{2}\max\{\lambda + \|x\|, 0\}\max\{-\mu - \alpha\|x\|, 0\}\}. \end{aligned}$$

However, from $z^+ \neq 0$ and $z^+ \in \Omega$, we get that $\lambda - \|x\| > 0$ or $\lambda + \|x\| > 0$. Thus, it follows from $(\lambda - \|x\|)(\mu - \alpha\|x\|) \leq 0$ and $(\lambda + \|x\|)(\mu + \alpha\|x\|) < 0$ that

$$\langle z^+, Tz^+ + q\rangle > 0.$$

This is a contradiction. Hence we prove that T has the row sufficiency property.

- Consider the case of $z^+ = 0$. In this case, we have $z^- \neq 0$. Define $q = -Tz^- + (T^*z)^+$. In a similar way as the above proof, we can verify that $(z^-, 0)$ is a KKT point of (QP) and z^- operator commutes with T^*z^-. Hence, z^- is a solution of LCP(T, Ω, q) and $\langle z^-, Tz^- + q\rangle = \langle z^-, (T^*z)^+\rangle = 0$. Furthermore, in a similar way again as in the case of $z^+ \neq 0$, we can obtain that $\langle z^-, (T^*z)^+\rangle > 0$, which is a contradiction. Hence, T has the row sufficiency property.

The proof is complete. □

Remark In fact, the representations of e_1 and e_2, given in Case 2 of the proof of Theorem 3.2, is a Jordan frame representation for the spin algebra, as those in the case of finite-dimensional Euclidean space R^n (see, for example, [1]).

4 The column-sufficiency of T

Recall that a linear transformation $T: H \to H$ is said to have the *cross commutative property* if for any $q \in H$ and any two solutions z_1 and z_2 of LCP(T, Ω, q), it follows that z_1 operator commutes with w_2 and z_2 operator commutes with w_1, where $w_i = Tz_i + q$ $(i = 1, 2)$. In this section, for a linear transformation $T : H \to H$, we will show that the column-sufficiency along with the cross commutative property is equivalent to the convexity of solution set of LCP(T, Ω, q) (if the solution set is nonempty).

Theorem 4.1 *For the bounded linear transformation T on H, the following statements are equivalent:*

- (a) *T has the column sufficiency property and the cross commutative property;*
- (b) *For any $q \in H$, if the solution set of LCP(T, Ω, q) is nonempty, the solution set is convex.*

Proof $(a) \Rightarrow (b)$: When LCP(T, Ω, q) has only one solution, the convexity of the solution set of LCP(T, Ω, q) is obvious. When LCP(T, Ω, q) has more than one solution, suppose that z_1 and z_2 are two distinct solutions of LCP(T, Ω, q) and $w_i = Tz_i + q$ $(i = 1, 2)$. According to the cross commutative property, we have $z_1(z_2)$ operator commutes with $w_2(w_1)$. By Lemma 2.4, it follows that $z_1 \circ w_2 \in \Omega$ and $z_2 \circ w_1 \in \Omega$. Define $z = z_1 - z_2$, then

$$-z \circ Tz = -(z_1 - z_2) \circ (w_1 - w_2) = (z_1 \circ w_2 + z_2 \circ w_1) \in \Omega.$$

Since z_i operator commutes with w_j $(i, j = 1, 2)$, it follows that z and Tz operator commute. By the column sufficiency property of T, we have $z \circ Tz = 0$, which implies that $\langle z, Tz\rangle = 0$, i.e.,

$$\langle z_1 - z_2, T(z_1 - z_2)\rangle = \langle z_1 - z_2, w_1 - w_2\rangle = -(\langle z_1, w_2\rangle + \langle z_2, w_1\rangle) = 0.$$

Since Ω is a self-dual cone, it follows that $\langle z_1, w_2\rangle = \langle z_2, w_1\rangle = 0$. For any $t \in [0, 1]$, let $u = tz_1+(1-t)z_2$. Since Ω is convex, we have $u \in \Omega$ and $Tu+q = tTz_1+(1-t)Tz_2+q = tw_1 + (1-t)w_2 \in \Omega$. Moreover,

$$\begin{aligned}\langle u, Tu + q\rangle &= \langle tz_1 + (1-t)z_2, tw_1 + (1-t)w_2\rangle \\ &= t^2\langle z_1, w_1\rangle + t(1-t)\langle z_1, w_2\rangle + t(1-t)\langle z_2, w_1\rangle + (1-t)^2\langle z_2, w_2\rangle \\ &= 0.\end{aligned}$$

Thus, u is a solution of LCP(T, Ω, q). This implies that the solution set of LCP(T, Ω, q) is convex.

$(b) \Rightarrow (a)$: When LCP(T, Ω, q) has only one solution. the cross commutative property can be easily verified. When LCP(T, Ω, q) has more than one solution for some $q \in H$, suppose that z_1 and z_2 are any two distinct solutions of LCP(T, Ω, q) and $w_i = Tz_i + q$ $(i = 1, 2)$. Since the solution set of LCP(T, Ω, q) is convex, for any $t \in (0, 1)$, we have $z = tz_1 + (1 - t)z_2$ is a solution of LCP(T, Ω, q). Let $w = T(tz_1 + (1-t)z_2) + q = tw_1 + (1-t)w_2$, we have

$$\begin{aligned}0 &= \langle z, w\rangle \\ &= t^2\langle z_1, w_1\rangle + t(1-t)(\langle z_1, w_2\rangle + \langle z_2, w_1\rangle) + (1-t)^2\langle z_2, w_2\rangle \\ &= t(1-t)(\langle z_1, w_2\rangle + \langle z_2, w_1\rangle).\end{aligned}$$

From $z_i, w_i \in \Omega$ $(i = 1, 2)$, it follows that $\langle z_1, w_2\rangle = \langle z_2, w_1\rangle = 0$. By Lemma 2.2 (i), this implies that T is cross commutative.

Next, we prove that T has the column sufficiency property. Suppose it is not the case. Then there exists $z \in H$ such that z and Tz operator commute and $-z \circ Tz \in \Omega$, but $z \circ Tz \neq 0$. Let $z = x+\lambda e$ and $Tz = y+\mu e$ with $x, y \in \langle e \rangle^{\perp}$ and $\lambda, \mu \in R$. Since z and Tz operator commute, there exists $\alpha \in R$ such that $y = \alpha x$ or $x = \alpha y$. Without loss of generality, we only consider the case of $y = \alpha x$. we discuss the following two subcases:

(I) When $x = 0$, we have $z = \lambda e$ and $-z \circ Tz = -\lambda\mu e \in \Omega$. By $z \circ Tz \neq 0$, we get that $\lambda \neq 0$ and $\mu \neq 0$. If $\lambda > 0$, then $-Tz = -\mu e \in \Omega$. Let $q = -Tz$. we can conclude that z and 0 are solutions to LCP(T, Ω, q). By the convexity of the solution set of LCP(T, Ω, q), for any $t \in (0, 1)$, tz is also a solution of LCP(T, Ω, q), i.e.,

$$tz \in \Omega, \quad T(tz) + q \in \Omega, \quad \text{and} \quad \langle tz, T(tz) + q\rangle = 0,$$

which leads to $t^2\langle z, Tz\rangle = -t\langle z, q\rangle = t\lambda\mu = t\langle z, Tz\rangle$. It follows that $t\lambda\mu = \lambda\mu$. This is a contradiction. Similarly, for the case $\lambda < 0$, let $q = Tz$. We can obtain that $-z$ and 0 are two solutions to LCP(T, Ω, q). As before, it follows that $t\lambda\mu = \lambda\mu$ for any $t \in (0, 1)$, which is also a contradiction. These demonstrate that T has the column sufficiency property.

(II) When $x \neq 0$, let $e_1 = \frac{e-\frac{x}{\|x\|}}{2}$ and $e_2 = \frac{e+\frac{x}{\|x\|}}{2}$. We replace T^* with T and proceed as in Case 2 in the proof of Theorem 3.2. Then $z = z^+ - z^-$ and $Tz = (Tz)^+ - (Tz)^-$. We define $q = (Tz)^+ - T(z^+)$. Because $Tz = T(z^+ - z^-) = T(z^+) - T(z^-)$, we get that $q = (Tz)^- - T(z^-)$. It is easy to verify that both z^+ and z^- are solutions to LCP(T, Ω, q). Since the solution set of LCP(T, Ω, q) is convex, we have $\langle \frac{z^++z^-}{2}, T(\frac{z^++z^-}{2}) + q\rangle = 0$, which implies that $\langle z^+, T(z^-) + q\rangle = \langle z^-, T(z^+) + q\rangle = 0$. This leads to

$$\langle z, Tz\rangle = \langle z^+ - z^-, (T(z^+) + q) - (T(z^-) + q)\rangle = 0.$$

However, from $-z \circ Tz \in \Omega$ and $z \circ Tz \neq 0$, it follows that $\langle z, Tz \rangle = \langle z \circ Tz, e \rangle < 0$. This is a contradiction.

Therefore, we have verified that T has the column sufficiency property. □

Remark The cross commutative property in Theorem 4.1 is indispensable. This can be illustrated by the following example.

Example Let S^2 be the set of all 2×2 real symmetric matrices. For any $X, Y \in S^2$, define the inner product and norm by

$$\langle X, Y \rangle = \text{Trace}(XY) \ \text{ and } \ \|X\| = \sqrt{\langle X, X \rangle} = \sqrt{\text{Trace}(XX)}.$$

Then, it is easy to prove that S^2 is a real Hilbert space. In this setting, let

$$e := \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix},$$

then $\|e\| = 1$. In addition, the Lorentz cone Ω in the Hilbert space S^2 determined by e can be written as

$$\Omega = \{X = x + \lambda e \in S^2 \mid x \in < e >^{\perp} \text{ and } \lambda \in R \text{ with } \lambda \geq \|x\|\},$$

where $\lambda = \langle X, e \rangle = \text{Trace}(Xe)$ and $\langle x, e \rangle = \text{Trace}(xe) = 0$. For any $X, Y \in S^2$, Jordan product is defined by

$$X \circ Y = (x + \lambda e) \circ (y + \mu e) = \mu x + \lambda y + \langle X, Y \rangle e.$$

In fact, it is easy to prove that $X \circ Y = \frac{XY+YX}{2\sqrt{2}}$. From this, together with [8] or [11], we conclude that X and Y operator commute if and only if $XY = YX$. For an $M \in R^{2\times 2}$, we define the Lyapunov transformation $L_M : S^2 \to S^2$ by

$$L_M(X) = \frac{1}{\sqrt{2}}(MX + XM^T).$$

The linear transformation T is said to have the P-property if

$$\left.\begin{array}{l} z \text{ and } Tz \text{ operator commute} \\ z \circ Tz \leq 0 \end{array}\right\} \quad \Longrightarrow \quad z = 0.$$

Similar to [11, Theorem 5], it is easy to show that the Lyapunov transformation L_M has the P-property if and only if M is positive stable. We consider the following cone linear complementarity problem LCP(L_M, Ω, Q): find a matrix $X \in S^2$ such that

$$X \in \Omega, \quad Y := L_M(X) + Q \in \Omega, \quad \text{and} \quad \langle X, Y \rangle = \text{Trace}(XY) = 0, \tag{4.1}$$

where

$$M := \begin{bmatrix} -3 & 5 \\ -5 & 5 \end{bmatrix} \ \text{ and } \ Q := \begin{bmatrix} 3\sqrt{2} & 5/\sqrt{2} \\ 5/\sqrt{2} & 9\sqrt{2} \end{bmatrix}.$$

It follows that $Q \in \Omega$ and M is positive stable, i.e., L_M has the P-property. Hence, L_M has the column sufficiency property. Moreover, it is easy to verify that

$$0 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \ \text{ and } \ N := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

are two solutions of LCP(L_M, Ω, Q) (4.1), but N doesn't operator commute with $W = L_M(0) + Q = Q$, i.e., L_M has no the cross commutative property. It is evident that, for any $\alpha \in (0, 1)$, αN is not a solution to LCP(L_M, Ω, Q). This demonstrates that the solution set of LCP(L_M, Ω, Q) is not convex.

5 Concluding remarks

In this paper, we introduced the concepts of the column-sufficiency and row-sufficiency for the bounded linear transformation T on the real (finite-dimensional or infinite-dimensional) Hilbert space H. After discussing several properties of z and w operator commuting, we established a necessary and sufficient condition of T being column sufficient and a necessary and sufficient condition of T being row sufficient. A further issue to be studied is to investigate how to solve the Lorentz cone complementarity problem on the Hilbert space.

Acknowledgments The authors are very grateful to the referee for his/her constructive comments and valuable suggestions, which have considerably improved the paper.

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